Superposition of Vectors and Applications in Coulomb's Law

1. Superposition of Vectors

The principle of superposition of vectors states that when two or more vectors act simultaneously at a point, their combined effect is given by their vector sum. The resultant vector is obtained by adding the individual vectors according to the rules of vector addition.

2. Techniques Under the Superposition of Vectors

Several techniques are used to determine the resultant vector.

a) Graphical Methods

These methods involve representing vectors geometrically.

i) Triangle Law of Vector Addition

Used when two vectors are added head-to-tail. The resultant is the third side of the triangle taken in the opposite order.

R = A + B

ii) Parallelogram Law of Vector Addition

If two vectors originate from the same point, they form a parallelogram when extended. The resultant is given by the diagonal of the parallelogram.

R = sqrt(A^2 + B^2 + 2AB cos θ)

b) Analytical Methods

These methods involve breaking vectors into components and using algebraic calculations.

i) Resolution of Vectors into Components

A vector can be resolved into perpendicular components along the x-axis and y-axis.

For a vector A with angle θ:
$A_x = A \cos\theta$
$A_y = A \sin\theta$

For multiple vectors:
$R_x = A_x + B_x + C_x + \ldots$
$R_y = A_y + B_y + C_y + \ldots$

The resultant vector:

$R = \sqrt{R_x^2 + R_y^2}$

Direction: $\theta = \tan^{-1}(R_y / R_x)$

c) Polygon Law of Vector Addition

When more than two vectors are added, they can be arranged in a polygon. The resultant is given by the closing side of the polygon.

d) Vector Addition Using Unit Vectors

Vectors can be expressed in terms of unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

If:
$A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$
$B = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$

Then the resultant:

$R = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} + (A_z + B_z) \hat{k}$

e) Superposition of Perpendicular Vectors

When vectors are perpendicular ($\theta = 90°$), their resultant simplifies to:
$R = \sqrt{A^2 + B^2}$

f) Dot and Cross Product

Dot Product: Finds the projection of one vector onto another.

$A \cdot B = AB \cos\theta$

Cross Product: Finds a vector perpendicular to both given vectors.

$A \times B = AB \sin\theta\, \hat{n}$

3. Finding the Direction of the Resultant Using the Parallelogram Law

The direction of the resultant vector is given by the angle α it makes with vector A:

$\tan\alpha = (B \sin\theta) / (A + B \cos\theta)$

4. Vector Form of Coulomb's Law

Coulomb's law in vector form for two charges q1 and q2 positioned at r1 and r2 is:

$F12 = k\,(q1\, q2 / |r12|^2)\, \hat{r}12$

where:

$r12 = r2 - r1$ is the displacement vector from q1 to q2,
$\hat{r}12 = r12 / |r12|$ is the unit vector in that direction.

5. Example: Three Charges in a Triangle

Problem Statement:

Three charges form an equilateral triangle of side 3m:

q1 = +4 μC at A,
q2 = +4 μC at B,
q3 = -4 μC at C.

Find the net force on q1 using vector notation.

Step 1: Define Position Vectors

$rA = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$
$rB = 3\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$
$rC = 1.5\,\hat{i} + 2.598\,\hat{j} + 0\,\hat{k}$

Step 2: Compute Displacement Vectors

$rAB = 3\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$
$rAC = 1.5\,\hat{i} + 2.598\,\hat{j} + 0\,\hat{k}$

Step 3: Compute Unit Vectors

$\hat{r}AB = 1\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$
$\hat{r}AC = 0.5\,\hat{i} + 0.866\,\hat{j} + 0\,\hat{k}$

Step 4: Compute Forces

$F_{AB} = 16 \times 10^{-3}$ N, $F_{AC} = 16 \times 10^{-3}$ N
$F_{AB} = 16 \times 10^{-3}\,\hat{i}$
$F_{AC} = 8 \times 10^{-3}\,\hat{i} + 13.86 \times 10^{-3}\,\hat{j}$

Step 5: Compute Net Force

$F_{net} = (16 + 8) \times 10^{-3}\,\hat{i} + (0 + 13.86) \times 10^{-3}\,\hat{j}$
$F_{net} = 24 \times 10^{-3}\,\hat{i} + 13.86 \times 10^{-3}\,\hat{j}$

Step 6: Compute Magnitude and Direction

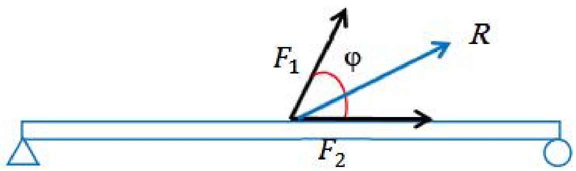
$F_{net} = \sqrt{(24 \times 10^{-3})^2 + (13.86 \times 10^{-3})^2} = 0.0277$ N
$\theta = \tan^{-1}(13.86 / 24) = 30°$

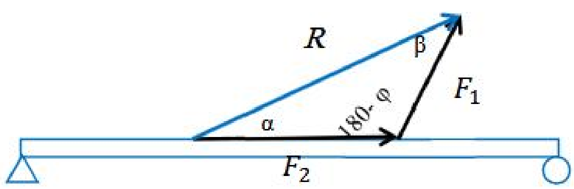
Conclusion

The net force on q1 is 0.0277 N at 30° below the x-axis.

Parallelogram Law

If F_1 and F_2 represent two forces and φ represents the angle between them, their effects can be replaced by the effect of on force, which called the resultant of forces (R) and as follows:



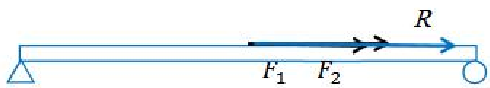


Then

$$R = \sqrt{{F_1}^2 + {F_2}^2 + 2F_1F_2\cos\varphi}$$

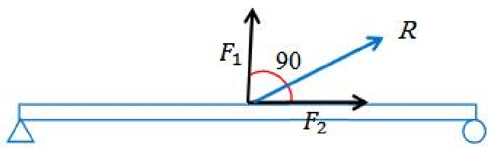
Apply triangle law:

$$\frac{R}{\sin(180-\varphi)} = \frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta}$$



$$R = \sqrt{{F_1}^2 + {F_2}^2 + 2F_1F_2\cos 0} = F_1 + F_2$$

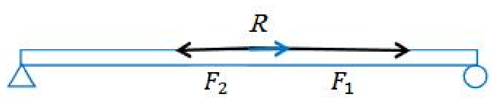
Case 2, the angle φ between F_1 and F_2 equal to 90°, then:



$$R = \sqrt{{F_1}^2 + {F_2}^2 + 2F_1F_2\cos 90} = \sqrt{{F_1}^2 + {F_2}^2}$$

$\alpha = \tan^{-1}(\frac{F_1}{F_2})$ the angle between R and F_2

Case 3, the angle φ between F_1 and F_2 equal to 180°, then:



$$R = \sqrt{{F_1}^2 + {F_2}^2 + 2F_1F_2\cos 180} = \sqrt{{F_1}^2 + {F_2}^2 - 2F_1F_2} = \sqrt{(F_1 - F_2)^2}$$
$$= F_1 - F_2$$

Apply triangle law:

$$\frac{R}{\sin(180-\alpha)} = \frac{Q}{\sin\varphi} = \frac{P}{\sin\beta}$$

$$\frac{1734.9}{\sin(180-60)} = \frac{1100}{\sin\varphi}$$

$$\varphi = \sin^{-1}\left(\frac{1100}{\frac{1734.9}{\sin(180-60)}}\right) = 33.3^{o}$$

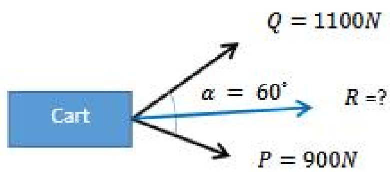
$$\frac{1734.9}{\sin(180-60)} = \frac{900}{\sin\beta}$$

$$\beta = \sin^{-1}\left(\frac{900}{\frac{1734.9}{\sin(180-60)}}\right) = 26.7^{o}$$

Problem1:

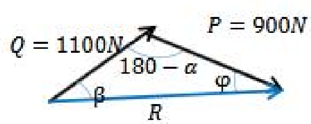
A cart is pulled uniformly along two cables using two horses as shown below, the tension forces along the two cables are $P = 900\ N$ and $Q = 1100\ N$ and the angle between them are $\alpha = 60^{\circ}$.

Determine the magnitude of the resultant force (R)and the angles between the resultant and the two tension forces.



Solution:

Using the Parallelogram Law:



$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha} = \sqrt{900^2 + 1100^2 + 2 \times 900 \times 1100 \times \cos 60}$$
$$= 1734.9N = 1.7349KN$$